MATH 2050 - Consequences of the Completeness Property

(Reference: Bartle § 2.4)

Completeness Property: Every $\phi \neq S \subseteq \mathbb{R}$ which is bounded above must have a supremum in $\mathbb{R}$.

Archimedean Property: $\mathbb{N}$ is NOT bdd above.

Pf: Suppose NOT, i.e. $\mathbb{N}$ is bdd above.

By Completeness Property, $\sup \mathbb{N} =: u \in \mathbb{R}$ exists.

So, $u - 1 < n'$ for some $n' \in \mathbb{N}$.

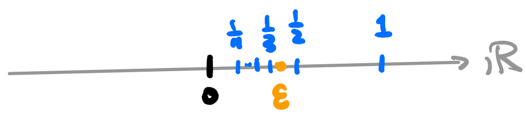
$\Rightarrow \quad u < n' + 1 \in \mathbb{N}$

$\Rightarrow \quad u$ is NOT an upper bd for $\mathbb{N}$ ← Contradiction!

$\square$

Corollaries:

(i) $\inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = 0$



(ii) $\forall \, \varepsilon > 0, \; \exists \, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \varepsilon$

(iii) $\forall \, y > 0, \; \exists! \, n \in \mathbb{N}$ st. $n - 1 < y \leq n$

unique

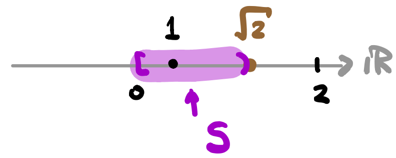
Ex: Prove these!

Recall: $\sqrt{2}$ & $\mathbb{Q} \subsetneq \mathbb{R}$

Thm: (Existence of $\sqrt{2}$ in $\mathbb{R}$)

$\exists\ x \in \mathbb{R}$ st. $x > 0$ and $x^2 = 2$.

Picture:



Proof: Let $S := \{ s \in \mathbb{R} : s \geq 0,\ s^2 < 2 \}$

Claim 1: $S \neq \phi$ $\quad (\because 0 \in S)$

Claim 2: S is bdd above.

Why? $\forall s \in S$, $s \geq 0$ and " $s^2 < 2 < 4 = 2^2 \overset{s \geq 0}{\Rightarrow} s < 2$ "

i.e. 2 is an upper bd for S

By Completeness Property, $x := \sup S \in \mathbb{R}$ exists.

* Claim 3: $x > 0$ and $\boxed{x^2 = 2.}$

Since $1 \in S$, and x is an upper bd for S.

$0 < 1 \leq x$ $\qquad$ Thus, $x > 0$.

To prove $x^2 = 2$, we argue by contradiction.

Suppose NOT, by Trichotomy, either $x^2 < 2$ OR $x^2 > 2$.

Case 1: $x^2 < 2$

WANT: Find $n \in \mathbb{N}$ s.t. $x + \frac{1}{n} \in S$

i.e. $\left(x + \frac{1}{n}\right)^2 < 2$.

This implies x is NOT an upper bd for S.

Contradicting $x = \sup S$.

$$\left(x+\frac{1}{n}\right)^2 < 2$$

$$\Uparrow$$

$$x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2$$

$$\Uparrow$$

$$x^2 + \frac{2x}{n} + \frac{1}{n} < 2$$

$$\Uparrow$$

$$\frac{2x+1}{n} < 2 - x^2$$

$$\Uparrow$$

$$\frac{1}{n} < \frac{2-x^2}{2x+1}$$

flow of proof

Now, by assumption $2 - x^2 > 0$.

also $x > 0 \Rightarrow 2x + 1 > 0$

Thus, $\dfrac{2-x^2}{2x+1} > 0$.

By Archimedean Property, $\exists\, n \in \mathbb{N}$ st.

$$0 < \frac{1}{n} < \frac{2-x^2}{2x+1} \quad \cdots\cdots (*)$$

Then, for this n,

$$\left(x+\frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$$

$$\left(\because \frac{1}{n^2} \le \frac{1}{n}\ \ \forall\, n \in \mathbb{N}\right) \le x^2 + \frac{2x}{n} + \frac{1}{n}$$

$$n \ge 1$$

$$= x^2 + \frac{2x+1}{n} < 2$$

by $(*)$

Case 2: $x^2 > 2$.

Want: Find $m \in \mathbb{N}$ s.t. $x - \frac{1}{m}$ is an upper bd for S

$(\Rightarrow x$ is NOT the least upper bd. contradicting $x = \sup S)$

Arch. Property

Choose $m \in \mathbb{N}$ s.t. $\dfrac{1}{m} < \dfrac{x^2-2}{2x}$ R.H.S. >0

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m} \ge 2 > s^2 \qquad \left(\because \frac{1}{m^2} > 0\right) \quad \forall\, s \in S$$

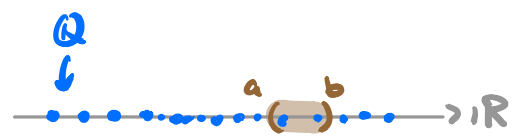
$\blacksquare$

Thm: (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For any $a, b \in \mathbb{R}$ s.t. $a < b$.

$\exists\, x \in \mathbb{Q}$ s.t. $a < x < b$.

Picture



Proof: Given $a, b \in \mathbb{R}$, $0 < a < b$. then $b - a > 0$.

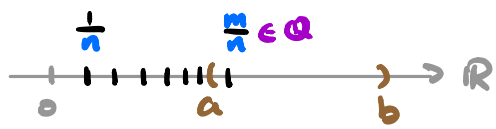
Step size

By Archimedean Property, $\exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < b - a$

Since $na > 0$, by Archimedean Property,

$\exists\, m \in \mathbb{N}$ s.t. $m - 1 \leq na < m$.

Picture:



Note: $\frac{1}{n} < b - a \Rightarrow na + 1 < nb$

$m - 1 \leq na < m \Rightarrow m \leq na + 1 < m + 1$

Combining these two inequalities,

$$na < m \leq na + 1 < nb$$

Divide by n $\Rightarrow$ $a < \frac{m}{n} < b$.

$\in \mathbb{Q}$

$\blacksquare$

Cor: $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$

Pf: Fix any $a, b \in \mathbb{R}$, want: $\exists\, y \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < y < b$.
$(a < b)$.

Consider $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ in $\mathbb{R}$, by density of $\mathbb{Q}$ in $\mathbb{R}$,

$\exists\, q \in \mathbb{Q}$ s.t. $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$

$$\Rightarrow a < \underbrace{q \cdot \sqrt{2}}_{\notin \mathbb{Q}} < b$$

$\blacksquare$